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What causes failure to apply the Pigeonhole Principle in simple reasoning problems?

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ABSTRACT
The Pigeonhole Principle states that if *n* items are sorted into *m* categories and if $n > m$, then at least one category must contain more than one item. For instance, if 22 pigeons are put into 17 pigeonholes, at least one pigeonhole must contain more than one pigeon. This principle seems intuitive, yet when told about a city with 220,000 inhabitants none of whom has more than 170,000 hairs on their head, many people think that it is merely likely that two inhabitants have the exact same number of hair. This failure to apply the Pigeonhole Principle might be due to the large numbers used, or to the cardinal rather than nominal presentation of these numbers. We show that performance improved both when the numbers are presented nominally, and when they are small, albeit less so. We discuss potential interpretations of these results in terms of intuition and reasoning.

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Introduction

The Pigeonhole Principle (also known as the box principle, the drawer principle or Dirichlet Principle, after the nineteenth century German mathematician, Gustav Lejeune Dirichlet who first formulated it) has wide and sophisticated mathematical applications (Mazur, 2010, pp. 40–48; Walker, 1977). Still, in its elementary form – the only one we are considering here – it is quite simple: If *n* items are sorted into *m* categories and if $n > m$, then at least one category must contain more than one item. For instance, if 22 pigeons are put into 17 pigeonholes, at least one pigeonhole must contain more than one pigeon.

The Pigeonhole Principle may seem rather intuitive, but is it really? There are a few well-known riddles that can be solved by simple application of the Principle but that baffle most people. The best known may be the same-

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number-of-hairs riddle (see for instance, Rignano, 1927, p. 73; Smullyan, 1978, pp. 8–9). Here is a version that we used in our experiments:

> In the city of Denton, there are 220,000 inhabitants. None of the inhabitants has absolutely no hair on his or her head. None of the inhabitants has more than 170,000 hairs on his or her head. How likely is it that at least two inhabitants of Denton have the exact same number of hair on their head?

If we categorise these 220,000 inhabitants by the number of hairs on their head, they fall into a maximum of 170,000 categories. Given that 220,000 $>$ 170,000, the Pigeonhole Principle applies: At least one category must contain more than one item. The correct answer, then, is that one can tell for sure that at least two inhabitants of Denton have exactly the same number of hair.

What makes the same-number-of-hairs problem an effective riddle and a problem of psychological relevance is that most people answer that one cannot tell for sure. Why does this riddle which, given the Pigeonhole Principle, can be simply solved, elicit such a mistaken answer?

To the best of our knowledge, the Pigeonhole Principle and reasoning mistakes that result from the failure to apply it have not been studied in the experimental psychology of reasoning. There is some indirectly relevant evidence in the psychology of education (e.g., Lax, 1999; Sriraman & Adrian, 2004). We believe that the issue is worth investigating (1) as robust example of a fallacious elementary inference and (2) because of its potential relevance to the study of the relationship between intuition and reasoning. Elementary Pigeonhole Principle problems may be comparable to the tasks used in the Cognitive Reflection Test (Frederick, 2005) and to similar problems discussed by Toplak, West, and Stanovich (2014), where people accept without reasoning an intuitive but incorrect response.

What explains most people's failure to solve the same-number-of-hairs problem? We consider two possible explanations:

(1) Sorting things into tens of thousands of categories is not something that people normally do or reason about doing. The difficulty, then, might be caused by the relatively large numbers involved in the hairs problem. If so, the difficulty should disappear in logically equivalent versions of the problem involving much smaller numbers.

(2) The difficulty may be caused by the facts that, in the hairs problem, numbers are explicitly used *cardinally* to represent quantity of hairs. Numbers of hairs can also be understood *nominally*, each possible number of hairs on a person's head identifying a different category of people: People with 1 hair, people with 2 hairs, … people with 170,000 hairs. It could be that thinking of numbers cardinally as representing quantities of hairs (as invited by the problem statement) stands in the

way of thinking of numbers nominally as representing categories, which is what should be done to solve the problem.

To test these two possible explanations, we presented participants with versions that differed along two dimensions: (1) Larger vs. smaller numbers (differing by a factor of 10,000), and (2) numbers used cardinally vs. numbers used nominally.

Method

Participants and design

A total of 198 participants were recruited through the Amazon Mechanical Turk website (69 females, $M_{age} = 29.54$, $SD = 9.53$). One hundred and seventy-four participants had at least some college education. They were paid $0.5 for their participation. All participants had to be in the US at the time of the experiment and gave their informed consent before engaging with the survey.

The design was a 2 × 2 between-participants. One variable was the size of the numbers used (High or Low), the other the way the numbers defining the relevant categories (the pigeonholes, so to speak) were explicitly used (cardinally or nominally).

Materials and procedure

Each participant saw one of the four following problems:

Cardinal High condition:

In the city of Denton, there are 220,000 inhabitants. None of the inhabitants has absolutely no hair on his or her head. None of the inhabitants has more than 170,000 hairs on his or her head. How likely is it that at least two inhabitants of Denton have the exact same number of hair on their head?

Nominal high condition:

In the city of Denton, there are 220,000 inhabitants. Every inhabitant has been given a lottery ticket. On every ticket, there is a number between 1 and 170,000. How likely is it that at least two inhabitants of Denton have the exact same number on their lottery ticket?

Cardinal low condition:

In the village of Denton, there are 22 farmers. All of the farmers have at least one cow. None of the farmers has more than 17 cows. How likely is it that at least two farmers in Denton have the exact same number of cows?

Nominal low condition:

In the village of Denton, there are 22 farmers. The farmers have all had a visit from the health inspector. The visits of the health inspector took place between the 1st and the 17th of February of this year. How likely is it that at least two farmers in Denton had the visit of the health inspector on the exact same day?

The answers offered were: *Certainly false*, *Probably false*, *Probably true* and *Certainly true*. After answering this question, the participants filled in demographic information. They were also asked if they had already encountered this problem before. No participant answered that they had.

Results

Figure 1 displays the percentage of participants providing each answer in the four conditions. Since we were investigating the conditions for correct answer, the following tests contrast correct answers (*Certainly true*) and the other three answers taken together (*Certainly false, Probably false* and *Probably true*). All the tests are two-tailed Fisher's exact tests. There was a significant effect of number type, with more correct answers in the nominal condition than in the cardinal condition (59% vs. 23%; $p < 0.0001$). This difference held true both for High numbers (47% vs. 16%; $p = 0.002$) and for Low numbers (70% vs. 30%; $p < 0.0001$). There was also a significant effect of number size, with more correct answers in the Low numbers condition than in the High numbers condition (50% vs. 32%; $p = 0.001$). This difference was significant in the nominal condition ($p = 0.025$), but not in the cardinal condition ($p = 0.153$).

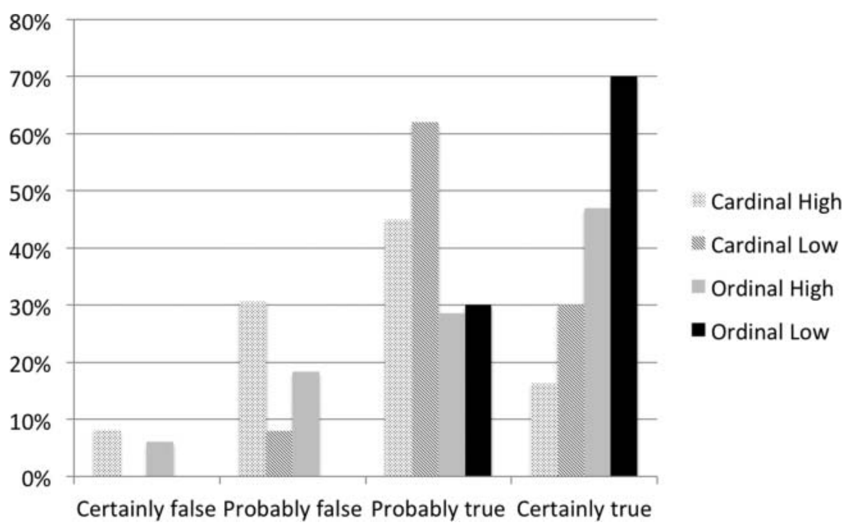

Figure 1. Percentage of participants providing each answer in the four conditions.

General discussion

When the Pigeonhole Principle is explained by means of the example that gives it its name (having a small number of pigeons fit into an even smaller number of pigeonholes), it seems quite intuitive. Why, then, does the same-number-of-hairs problem – which can be straightforwardly solved by applying this very principle – elicit a majority of mistaken responses? We considered two possible explanations: (1) The difficulty might be due to the relatively large numbers (220,000 and 170,000) involved in the hairs problem, or (2) it might be due to the fact that the second of these numbers is explicitly used to refer to the maximum possible number of hairs and not to the maximum number of categories of people categorised by their number of hairs.

While both factors do have an effect on participants' success, manipulating number size was not sufficient to render the problem easy, whereas manipulating the explicitly cardinal vs. nominal use of the relevant number was. In both the cardinal high and the cardinal low conditions, the modal answer was the incorrect *Probably true* answer. In both the nominal high and the nominal low condition, the modal answer was the correct *Certainly true* answer. (Note that, in the nominal high condition, it may have seemed odd to participants that several lottery tickets should have the same exact number and this could have induced them to avoid the answer *Certainly true*; nevertheless, this answer was given by 47% of the participants while only 29% answered *probably true*.) The nominal rather than cardinal explicit use of numbers, then, was the main factor of success or failure in answering the problem in its four versions.

These results are consistent with the following tentative conclusion: The Pigeonhole Principle is genuinely intuitive. People are capable of applying it spontaneously to a problem framed as one of sorting a larger number of items into a smaller number of categories, even when numbers of items and categories are as high as in the same-number-of-hairs problem.

For the Pigeonhole Principle intuitions to apply, however, the problem must be understood as one of sorting items into categories. This understanding is intuitive when numbers are used nominally to identify categories, even if only implicitly. When, on the other hand numbers are used cardinally to denote quantities (for instance of hairs or of cows) then the possibility of defining categories in terms of these numbers does not come to the mind of most participant. In such conditions, some proper reasoning is needed not to produce a direct solution to the problem, but to reframe it as one that elicits Pigeonhole Principle intuitions. Given their properties, problems relying on the Pigeonhole Principle could constitute a further tool to investigate the validity of dual process models of reasoning (e.g., Evans & Stanovich, 2013). In particular, the existence of easy (nominal) and hard (cardinal) versions of the problem might help investigate "logical intuitions" (for review, see De Neys, 2012): Are people who provide the incorrect answer to the hard version somehow conflicted, and thus less confident in their answers than those who provide the correct answer to the easy version?

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